TOPIC 16: TWO PERSON GAMES (OPTIMAL FIXED STRATEGY)

By inspecting the payoff matrix, we can sometimes rule out weak strategies if the payoff for another strategy is always better, no matter what the opponent does.

1. Dominated Strategies.

A strategy is said to be **dominated** by a second strategy if the second strategy always results in at least as good an outcome for the player, no matter what strategy the other player chooses, and results in a better outcome for at least one of the opponent's strategies. We call the inferior strategy a **dominated strategy.** A strategy which dominates all others is called a **dominant strategy**. It must be unique.

Example 1.1. *Recall our example from the previous section, where two fitness companies were trying to decide where to set up their new Gymnasiums. The payoff matrix is shown below*

		Fitness Indiana	
		First Neighborhood	**Second Neighborhood**
Get Up 'n Go	**First Neighborhood**	(1500, 3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

(a) Use the payoff matrix for the example above to determine if either Fitness Indiana or Get up 'n Go have any dominated strategies.

(b) Use the payoff matrix for the example above to determine if either Fitness Indiana or Get up 'n Go have any dominant strategies.

2. The Reduced Payoff Matrix and Equilibrium

Because there is always a better option than a dominated strategy, it would be very unwise of a player to select a dominated strategy. Therefore a player should eliminate all dominated strategies from his/her options, thus reducing the size of the matrix. After the dominated strategies for both players have been removed from the payoff matrix, some new dominated strategies may appear in the new partially reduced payoff matrix. After a series of reductions, we arrive at a matrix with no dominated strategies. This is called the **Reduced Payoff Matrix**.

Example Find the reduced payoff matrix for our example

		Fitness First Neighborhood	Indiana Second Neighborhood
Get Up 'n Go	**First Neighborhood**	(1500, 3500)	(5000, 3000)
	Second Neighborhood	(3000, 5000)	(900, 2100)

Definition 2.1. *The* **Reduced Payoff Matrix** *of a game is a submatrix of the game where dominated strategies have been eliminated in one or more stages. It gives the relevant portion of the original payoff matrix under the assumption of best play by both players.*

Note that in the above example, the reduced payoff matrix has just a single strategy for both players. For this strategy, neither player has any incentive to change strategy if the other player sticks with their current strategy. This is called an equilibrium point.

Definition 2.2. *An* **Equilibrium Point** *of a game is a pair of strategies such that neither player has any incentive to change strategies if the other player stays with their current strategy. For this reason it is* **Stable** *and once it is reached, it will generally persist through repeated playing of the game. When a game is not at an equilibrium point, at least one player has an incentive to change strategies, such a point is called* **unstable**. *A game may have a unique equilibrium point, more than one equilibrium point or no equilibrium points.*

Whenever elimination of dominant strategies leads to single strategies for both players, the pair of strategies form an equilibrium point. However, as shown below, equilibrium points may also occur in other ways.

Example 2.1. *In the example from Dutta* [1] *from the previous section, we found that the payoff matrices (where d denotes the strategy of taking the drug and n denotes the strategy of not taking the drug for each player) were different depending on whether the IOC performed a drug test (on just one player) or not. The pay off matrices are given below where b is a very large number:*

No Testing (Probabilities)

		Carter: **d**	Carter: **n**
Rogers	**d**	0.5	1
	n	0	0.5

IOC Testing (Expected Payoff)

		Carter: **d**	Carter: **n**
Rogers	**d**	$\left(-\frac{b}{2}, -\frac{b}{2}\right)$	$\left(-\frac{b}{2}, \frac{1}{2}\right)$
	n	$((\frac{1}{2}, -\frac{b}{2})$	$(\frac{1}{2}, \frac{1}{2})$

(a) Find the reduced payoff matrix for each scenario.

(b) Find the equilibrium points for each scenario.

Example 2.2 **(Two Equilibrium Points).** *Rose and Colin are playing a game where each has three strategies with payoff matrix is shown below:*

	C1	**C2**	**C3**
R1	(6. 4)	(7, 1)	(8, 6)
R2	(1, 2)	(9, 5)	(4, 7)
R3	(8, 8)	(6, 2)	(3, 3)

(a)Find the reduced payoff matrix for this game.

(b) Find the equilibrium points

2.1. **Dominated Strategies and Equilibrium points in a constant sum game.** For a constant sum game or a zero-sum game, we just write the payoff for the row player since we can deduce the payoff for the column player from it. In this case **a dominated strategy for the row player** corresponds to a row where the entries are less than or equal to the corresponding entries in another row (the dominating strategy). **A dominated strategy for the column player** corresponds to a column whose entries are greater than or equal (since the entries are payoffs for the row player) to the corresponding entries in another column (the dominating column).

Definition 2.3. *In the case of zero-sum or constant-sum games an equilibrium point is called a* **saddle point**. *The value of the pay-off matrix at that position is called the* **value** *of the game. If an equilibrium point exists in the game, it occurs at a point which is simultaneously the minimum in its row and the maximum in its column (since neither player has an incentive to change strategy at that point). Although the equilibrium point may not be unique, if there are multiple equilibrium points for the game, all will give the same value (payoff for the row player).*

To find equilibrium points for constant-sum games, we can strike out dominated strategies as above or we can calculate the minimum for each row and the maximum for each column and see if any entry in the matrix simultaneously gives the minimum in its row and the maximum in its column.

Example 2.3. *A (hypothetical) baseball pitcher throws three pitches, a fastball, a slider and a change-up. As a measure of the payoff for this type of confrontation, one might use the expected number of runs the batter creates in each situation. (Note their are other possible measures that take into account the pitcher's abilities). We would expect that for any given pitch, the batter's performance is better if he anticipates the pitch. Lets assume that the batter has four possible strategies, To anticipate either a fastball, a slider or a change-up or not to anticipate any pitch.*

		Pitcher		
		Fastball	**Change-up**	**Slider**
Batter	**Fastball**	0.3	0.3	0.35
	Change-up	0.25	0.4	0.4
	Slider	0.2	0.39	0.45
	None	0.3	0.39	0.4

(a) Is there a saddle point in the above matrix? If so where is it?

(b) Find the reduced payoff matrix for this example.

Example 2.4. *In the example from Winston* [2] *in the last section, the football team on offense had the options of running or passing the ball and the team on defense could choose a run defense or a pass defense. We got the following pay-off matrix for the team on offense using expected gain in yards for each situation:*

		Defense	
		Run Defense	**Pass Defense**
Offense	**Run**	-5	5
	Pass	10	0

(a) Does this matrix have a saddle point?

3. Choosing a Strategy in a zero-sum game

From now on, we will specialize to the case of two-person zero-sum (or constant-sum) simultaneous move games which we will just refer to as a "a zero-sum game".

We make the following assumptions about the players:

- Both wish to maximize their payoff,
- Each player has full knowledge of the payoff matrix,
- Their opponent will play intelligently and wishes to maximize their own payoff.

Note that in a zero sum game The column player maximizes their payoff by minimizing the row players payoff.

For a zero-sum game or a constant-sum game with a saddle point, if the above assumptions hold and the game is played repeatedly the play will eventually stabilize at the saddle point.

Definition 3.1. *A player is said to play a* **fixed strategy** *or a* **pure strategy** *if the player always plays the same row (for a row player) or column (for a column player).*

Definition 3.2. *For a zero-sum game or a constant-sum game, if an equilibrium point exists (at least one), then we say that the game is* **strictly determined**.

When a game is strictly determined we have:

- The best strategy for both players is a fixed strategy (as we will see below) with the row player playing at the row in which the saddle point occurs and the column player playing at the column in which the saddle point occurs.
- The value of the game is the long run expected payoff for R when the game is played repeatedly since neither player will have any incentive to choose a different strategy than the one at the saddle point.

Above, we saw that a saddle point does not always exist in a zero-sum game. In this case it turns out that a mixed strategy is better to maximize long run expected payoff. However, if for some reason the row player or the column player may wish to choose a fixed strategy in which case the **best fixed strategy** will be the fixed strategy which will maximize their long run expected payoff. We assume that the opponent will notice that a player is playing a fixed strategy when the game is played repeatedly and will respond accordingly. (This is certainly not an unrealistic assumption in sports, we noted that in one of the more recent papers on the hot hand, the author noted that defense increased around the player with the hot hand because players tend to give the ball to that player).

Assume we have two players, R and C, where R has m strategies $r_1, r_2, \ldots, r_m$ and C has n strategies $c_1, c_2, \ldots, c_n$ where the payoff matrix is the $m \times n$ matrix shown below:

C

		c_1	c_2	$\ldots$	c_n
	r_1	a_{11}	a_{12}	$\ldots$	a_{1n}
	r_2	a_{21}	a_{22}	$\ldots$	a_{2n}
R	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	r_m	a_{m1}	a_{m2}	$\ldots$	a_{mn}

3.1. Finding the optimal fixed strategy for R.

If R is going to play the same fixed strategy repeatedly, we can assume that C will minimize R's payoff by choosing the strategy (column) which corresponds to the minimum payoff (for R) in the row. Thus for each row of the payoff matrix, R can assume that their payoff will be the minimum entry in that row if they choose that strategy. To find the **Optimal fixed/pure strategy for R**:

(1) For each row of the pay-off matrix (R's Pay-off matrix), find the least element.

C

		c_1	c_2	$\dots$	c_n	Min
	r_1	a_{11}	a_{12}	$\dots$	a_{1n}	min of row 1
	r_2	a_{21}	a_{22}	$\dots$	a_{2n}	min of row 2
R	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	r_m	a_{m1}	a_{m2}	$\dots$	a_{mn}	min of row m

(2) Choose the row for which this element is as large as possible (the row corresponding to the maximum of the numbers in the right hand column above).

Example 3.1. *What is the optimal fixed strategy for Charlie in the game of two Finger Morra?*

Charlie

		$(1,2)$	$(1,3)$	$(2,3)$	$(2,4)$
R	$(1,2)$	0	2	-3	0
u	$(1,3)$	-1	0	0	3
t	$(2,3)$	3	0	0	-4
h	$(2,4)$	0	-3	4	0

3.2. **Finding the Optimal Fixed Strategy for** C**.** Similarly if C is searching for an optimal fixed strategy, they know that no matter which strategy (column) they settle on, the row player will choose the row which gives the maximum payoff for R. Thus to find the **Optimal fixed/pure strategy for C**:

(1) For each column of the pay-off matrix (R's Pay-off matrix), find the largest element.

C

		c_1	c_2	$\dots$	c_n
	r_1	a_{11}	a_{12}	$\dots$	a_{1n}
	r_2	a_{21}	a_{22}	$\dots$	a_{2n}
R	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	r_m	a_{m1}	a_{m2}	$\dots$	a_{mn}
	Max	max. of Col 1	max. of Col 2	$\dots$	max. of Col n

(2) Choose the column for which this element is as small as possible (the column corresponding to the minimum of the numbers in the bottom row above).

Example 3.2. *We saw that in the example from Winston* [2] *in the last section, the football team on offense had the options of running or passing the ball and the team on defense could choose a run defense or a pass defense. We got the following pay-off matrix for the team on offense using expected gain in yards for each situation:*

		Defense	
		Run Defense	**Pass Defense**
Offense	**Run**	-5	5
	Pass	10	0

(a) What is the optimal fixed strategy for the offense in this case?

(b) What is the optimal fixed strategy for the Defense in this case?

As mentioned above,for any zero-sum game without a saddle point, repeatedly using the same strategy is not advisable since a higher long run expected payoff can be achieved using a mixed strategy (an unpredictable mix of your strategies). We will study how to find the best mixed strategy in the next section. The general principles are the same as that for finding the best fixed strategy shown above. We can expect that our opponent will anticipate our actions and play the best counterstrategy minimizing our payoff. Thus we will choose the strategy which maximizes the minimum payoff (called the max/min solution).

References

1. Prajit K. Dutta, *Strategies and games*, The M.I.T. Press.
2. Wayne Winston, *Mathletics*, Princeton University Press.